

ON THE PHASE TRANSITION CURVE IN A DIRECTED EXPONENTIAL RANDOM GRAPH MODEL

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ABSTRACT. We consider a family of directed exponential random graph models parametrized by edges and outward stars. Essentially all of the statistical content of such models is given by the *free energy density*, which is an appropriately scaled version of the probability normalization. We derive precise asymptotics for the free energy density of *finite* graphs. We use this to rederive a formula for the limiting free energy density first obtained by Chatterjee and Diaconis [3]. The limit is analytic everywhere except along a phase transition curve first identified by Radin and Yin [18]. Building on their results, we carefully study the model along the phase transition curve. In particular, we give precise scaling laws for the variance and covariance of edge and outward star densities, and we obtain an exact formula for the limiting edge probabilities, both on and off the phase transition curve.

1. INTRODUCTION

Probabilistic ensembles with one or more adjustable parameters are often used to model complex networks, including social networks, biological networks, the Internet, etc.; see e.g. Fienberg [5, 6], Lovász [14] and Newman [15]. One of the standard complex network models is the exponential random graph model, originally studied by Besag [2]. We refer to Snijders et al. [21], Rinaldo et al. [19] and Wasserman and Faust [22] for history and a review of recent developments.

The phenomenon of phase transitions in exponential random graph models has recently attracted a lot of attention in the literature. The statistical content of such models can be described by the *free energy density*, an appropriately scaled version of the probability normalization. The free energy density is also a standard quantity in statistical physics. In particular, its limit as the system size becomes infinite can be used to draw phase diagrams corresponding (for example) to the familiar fluid, liquid and solid phases of matter [7]. Using the large deviations formula for Erdős-Rényi graphs of Chatterjee and Varadhan [4], Chatterjee and Diaconis [3] obtained a variational formula for the limiting free energy density for a large class of exponential random graph models. Radin and Yin [18] used this to formalize, for the first time, the notion of a phase transition for exponential random graphs, explicitly computing phase diagrams for a family of two-parameter models. A similar three-parameter family was studied by Yin [24]. Previous non-rigorous analysis using mean-field theory and other approximations can be found in Park and Newman [16, 17] and the references in Häggström and Jonasson [9].

Date: 25 April 2014. *Revised:* 25 April 2014.

2000 *Mathematics Subject Classification.* 05C80, 82B26.

Key words and phrases. dense random graphs, exponential random graphs, graphs limits, phase transitions.

We consider a family of directed exponential random graphs parametrized by edges and outward directed p -stars. Such models are standard and important in the literature of social networks, see e.g. Holland [10] and the references therein. For directed graphs, recently developed techniques based on the graph limit theory of Lovasz [13] and the large deviations formula of Chatterjee and Varadhan [4] cannot be directly applied. Instead of trying to adapt these techniques to the directed case, we use completely different methods which lead to *better* asymptotics for the free energy density. From the limiting free energy density, we find that the model has a phase diagram essentially identical to the one of [18]. Because our asymptotics are more precise, we are able to build on the results in [18]. In particular, by carefully studying partial derivatives of the free energy density along the phase transition curve, we obtain precise scaling laws for the variance of edge and star densities and we compute exactly the limiting edge probabilities.

To explain how our results fit into the phase transition framework of [18], we need to make the notions of free energy and phase transition more precise. Consider the probability measure on the set of graphs on n nodes defined by

$$\mathbb{P}_n(X) = Z_n(\beta_1, \beta_2)^{-1} \exp(n^2 [\beta_1 e(X) + \beta_2 s(X)]), \quad (1.1)$$

where β_1, β_2 are real parameters, $Z_n(\beta_1, \beta_2)$ is the probability normalization, and $e(X)$ (resp. $s(X)$) is the probability that a random function from a single edge (resp. a p -star) into X is a homomorphism, i.e., an edge preserving map between the vertex sets. The quantities $e(X)$ and $s(X)$ are called homomorphism densities; see e.g. [3] for details and a discussion. We consider both undirected and directed graphs.

The model (1.1) has at least a superficial similarity to the grand canonical ensemble in statistical physics, which describes the statistical properties of matter in thermal equilibrium [8]. The grand canonical ensemble consists of a probability measure, defined on the set of locally finite subsets of $[-n/2, n/2]^d$, of the form

$$\mathbb{P}_n(Y) = Z_n(\beta, \mu)^{-1} \exp(n^d [-\beta \mu N(Y) - \beta E(Y)]), \quad (1.2)$$

where $\beta = 1/(k_B T)$ (with T temperature and k_B Boltzmann's constant), μ is chemical potential, $N(Y) = |Y|/n^d$ is the number density of Y , $E(Y)$ is the energy density of Y , and $d = 2$ or 3 . Here, each point of Y represents a particle (e.g., atom). A standard fact in statistical physics is that essentially all relevant physical quantities in the model can be obtained through the *free energy density*

$$\psi_n(\beta, \mu) := n^{-d} \log Z_n(\beta, \mu).$$

In particular, the average and variance of $N(Y)$ and $E(Y)$ (or more generally, all of their moments) can be obtained by differentiating ψ_n with respect to β or μ . Moreover, under appropriate conditions on E the limit

$$\psi(\beta, \mu) := \lim_{n \rightarrow \infty} \psi_n(\beta, \mu)$$

exists, and for any $i, j \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \frac{\partial^{i+j}}{\partial \beta^i \partial \mu^j} \psi_n(\beta, \mu) = \frac{\partial^{i+j}}{\partial \beta^i \partial \mu^j} \lim_{n \rightarrow \infty} \psi_n(\beta, \mu) = \frac{\partial^{i+j}}{\partial \beta^i \partial \mu^j} \psi(\beta, \mu)$$

whenever the derivative on the right hand side exists [23]. The limit $\psi(\beta, \mu)$ is key to understanding phases of matter: for instance, for a “typical” energy density E (e.g., based on the commonly used Lennard-Jones particle interaction [12]), that the function ψ is analytic except along two curves with an endpoint; these curves

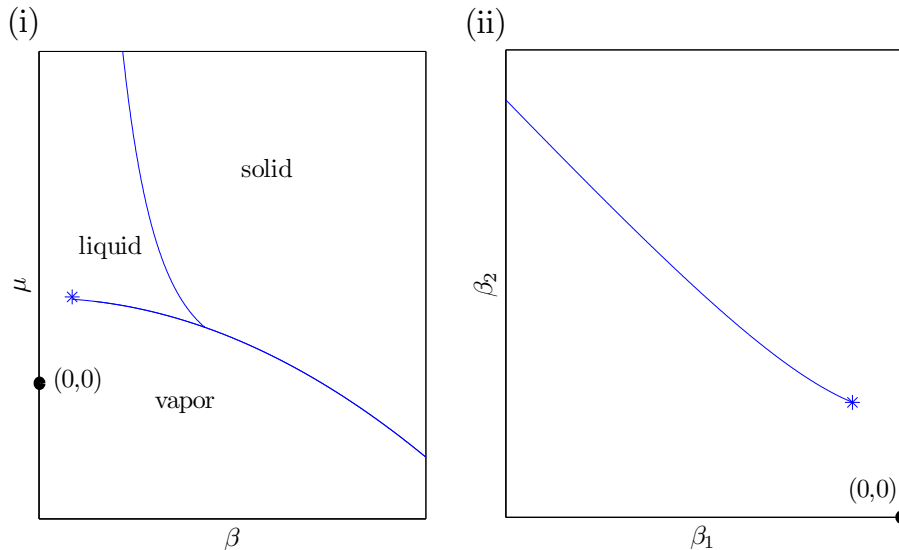


FIGURE 1. Simple phase diagrams in (i) the grand canonical ensemble, and (ii) the ERGM model. The critical point is labeled with a *.

correspond exactly to the solid/liquid/vapor phase transitions, and the endpoint is called the critical point [7]. See Figure 1(i). Actually, though the preceding statement is widely believed and supported by numerical experiments, proofs are possible only in very special cases; see e.g. Lebowitz et. al. [11].

The following analogy between the models (1.1) and (1.2) was explored in Radin and Yin [18]. For the model (1.1) we can define the free energy density in the same way,

$$\psi_n(\beta_1, \beta_2) = n^{-2} \log Z_n(\beta_1, \beta_2).$$

It is proved in [18] that in the undirected graph case, the limit

$$\psi(\beta_1, \beta_2) = \lim_{n \rightarrow \infty} \psi_n(\beta_1, \beta_2)$$

is analytic except along a certain curve with an endpoint, which we will call the phase transition curve and critical point, respectively; see Figure 1(ii). Moreover, on the curve but away from the critical point, the first order partial derivatives of ψ have a jump discontinuity; at the critical point, the first order partial derivatives of ψ are continuous but the second order derivatives diverge. Precisely the same behavior occurs on the liquid-vapor transition curve in the model (1.2).

To understand these singularities better, consider the following. First, just as in the grand canonical ensemble (1.2),

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\partial}{\partial \beta_i} \psi_n(\beta_1, \beta_2) &= \frac{\partial}{\partial \beta_i} \psi(\beta_1, \beta_2), \quad i \in \{1, 2\} \\ \lim_{n \rightarrow \infty} \frac{\partial^2}{\partial \beta_i \partial \beta_j} \psi_n(\beta_1, \beta_2) &= \frac{\partial^2}{\partial \beta_i \partial \beta_j} \psi(\beta_1, \beta_2), \quad i, j \in \{1, 2\}, \end{aligned}$$

provided the derivatives on the right hand side exist [18]. Next, straightforward computations show that

$$\begin{aligned} \frac{\partial}{\partial \beta_1} \psi_n(\beta_1, \beta_2) &= \mathbb{E}_n[e(X)], & \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) &= n^2 \text{Var}_n(e(X)) \\ \frac{\partial}{\partial \beta_2} \psi_n(\beta_1, \beta_2) &= \mathbb{E}_n[s(X)], & \frac{\partial^2}{\partial \beta_2^2} \psi_n(\beta_1, \beta_2) &= n^2 \text{Var}_n(s(X)). \end{aligned} \quad (1.3)$$

Thus, a jump discontinuity in $\partial\psi/\partial\beta_1$ (resp. $\partial\psi/\partial\beta_2$) along the transition curve implies a jump in the average value of $e(X)$ (resp. $s(X)$) across the curve as $n \rightarrow \infty$. Similarly, at the critical point, divergence of $\partial^2\psi/\partial\beta_1^2$ (resp. $\partial^2\psi/\partial\beta_2^2$) implies that the variance of $e(X)$ (resp. $s(X)$) decays more slowly than n^{-2} . Away from the transition curve, all partial derivatives of ψ (of any order) exist and are finite, so in particular the variance of $e(X)$ and $s(X)$ decays at least as fast as n^{-2} . Analogous statements can be made in the model (1.2) about the average and variance of $N(Y)$ and $E(Y)$. More detailed statements would require an analysis of the free energy density ψ_n for *finite* n ; this is much more difficult to study than the limit ψ , which be obtained (at least in the undirected case) via the large deviations results of Chatterjee and Diaconis [3].

In this paper we consider (1.1) in the directed graph case, with H_2 an outward directed p -star. For finite n , we obtain asymptotics for ψ_n and certain quantities related to its partial derivatives. Besides using these asymptotics to rederive a formula essentially the same as the one in [3, 18] for the limiting free energy density ψ , we obtain precise scaling of the variance and covariance of $e(X)$ and $s(X)$ as $n \rightarrow \infty$ for all parameter values (β_1, β_2) , including on the transition curve and at the critical point. By analogy with the model (1.2), the scaling at the critical point yields what in physics is called the *critical exponent* [8]. We also use our asymptotics for ψ_n to prove that in the limit $n \rightarrow \infty$, there is an edge between fixed nodes according to a Bernoulli random variable. On the transition curve, across which we recall the average of $e(X)$ has a jump discontinuity in the limit $n \rightarrow \infty$, we give an explicit formula for the Bernoulli parameter as a convex combination of the averages of $e(X)$ on both sides of the jump.

The paper is organized as follows. In Section 2, we describe the model in detail. Main results are stated in Section 3. The results are obtained by estimates, stated in Section 4, which allow for a precise computation of the free energy density and derivatives thereof. All proofs are in Section 5.

2. DESCRIPTION OF THE MODEL

Consider directed graphs on n nodes, where a graph is represented by a matrix $X = (X_{ij})_{1 \leq i, j \leq n}$ with each $X_{ij} \in \{0, 1\}$. Here, $X_{ij} = 1$ means there is a directed edge from node i to node j ; otherwise, $X_{ij} = 0$. Give the set of such graphs the probability

$$\mathbb{P}_n(X) = Z_n(\beta_1, \beta_2)^{-1} \exp \left[n^2 (\beta_1 e(X) + \beta_2 s(X)) \right], \quad (2.1)$$

where

$$\begin{aligned} e(X) &:= n^{-2} \sum_{1 \leq i, j \leq n} X_{ij}, \\ s(X) &:= n^{-p-1} \sum_{1 \leq i, j_1, j_2, \dots, j_p \leq n} X_{ij_1} X_{ij_2} \cdots X_{ij_p}. \end{aligned} \quad (2.2)$$

Here, $Z_n(\beta_1, \beta_2)$ is the appropriate normalization. Note that $e(X)$ and $s(X)$, defined in (2.2), represent the directed edge and outward directed p -star homomorphism densities of X . It is easy to see that $s(X)$ has the alternative expression

$$s(X) = n^{-p-1} \sum_{i=1}^n \left(\sum_{j=1}^n X_{ij} \right)^p. \quad (2.3)$$

We allow X_{ii} to equal 1 for ease of notation. It is not hard to see that without this simplification, our main results in Section 3 below hold exactly as stated, and our estimates in Section 4 hold with only small modifications.

Define the free energy density

$$\psi_n(\beta_1, \beta_2) := n^{-2} \log Z_n(\beta_1, \beta_2)$$

and the limiting free energy density

$$\psi(\beta_1, \beta_2) = \lim_{n \rightarrow \infty} \psi_n(\beta_1, \beta_2).$$

Our analysis will involve a closely related function

$$\ell(x) := \beta_1 x + \beta_2 x^p - x \log x - (1-x) \log(1-x).$$

It is easy to see that ℓ is analytic in $(0, 1)$ and continuous on $[0, 1]$. Note that ℓ is essentially identical to the function of the same name studied in [18]: after multiplying β_1 and β_2 by two the functions differ only by a constant. This allows us to use results from [18] concerning ℓ .

3. MAIN RESULTS

We rederive the following formula for the limiting free energy density, first obtained in [3] in the undirected graph case:

Theorem 1. *For any β_1, β_2 we have*

$$\psi_n(\beta_1, \beta_2) = \ell(x^*) + O(n^{-1} \log n).$$

We restate the following result proved in [18]:

Theorem 2 (Radin and Yin [18]). *There is a certain curve in the (β_1, β_2) -plane with the endpoint*

$$(\beta_1^c, \beta_2^c) = \left(\log(p-1) - \frac{p}{p-1}, \frac{p^{p-1}}{(p-1)^p} \right),$$

such that off the curve and at the endpoint, ℓ has a unique global maximizer $x^ \in (0, 1)$, while on the curve away from the endpoint, ℓ has two global maximizers, x_1^* and x_2^* , with $0 < x_1^* < (p-1)/p < x_2^* < 1$.*

The curve in Theorem 2 will be called the *phase transition curve* and written $\beta_2 = q(\beta_1)$. The endpoint will be called the *critical point*. It is not possible to write an explicit equation for the curve; see [18] for a graph obtained numerically. However, in [18] it is shown that $q(\beta_1)$ is continuous and decreasing in β_1 , with $\lim_{\beta_1 \rightarrow -\infty} |q(\beta_1) + \beta_1| = 0$. We have the following more precise result, which, since it concerns only the function ℓ , holds for both our model and that of [18]:

Theorem 3. (i) $q(\beta_1)$ is differentiable for $\beta_1 < \beta_1^c$ with

$$q'(\beta_1) = -\frac{x_1^* - x_2^*}{(x_1^*)^p - (x_2^*)^p} < 0.$$

In particular,

$$\lim_{\beta_1 \rightarrow \beta_1^c} q'(\beta_1) = -\frac{p^{p-2}}{(p-1)^{p-1}}, \quad \text{and} \quad \lim_{\beta_1 \rightarrow -\infty} q'(\beta_1) = -1.$$

(ii) $q(\beta_1)$ is convex in β_1 .

When $p = 2$, along the line $\beta_1 + \beta_2 = 0$ the function ℓ is symmetric around $1/2$. It follows that $x_1^* + x_2^* = 1$ along this line, so Theorem 3 implies $\ell(\beta_1) = -\beta_1$. See Figure 2(i).

As discussed in the introduction, the following theorems give the scaling of the variance of $e(X)$ and $s(X)$. Note that we compute this for any (β_1, β_2) , including on the phase transition curve and at the critical point; this extends the results in [18], which only hold off the phase transition curve.

Theorem 4. Off the phase transition curve,

$$\lim_{n \rightarrow \infty} \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) = \frac{\partial^2}{\partial \beta_1^2} \lim_{n \rightarrow \infty} \psi_n(\beta_1, \beta_2) = \frac{1}{|\ell''(x^*)|}.$$

On the phase transition curve except at the critical point,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) = \frac{(x_1^* - x_2^*)^2 \sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}{\left(\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} + \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|} \right)^2}.$$

At the critical point,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{2\sqrt{6}(p-1)}{p^{5/2}}.$$

Theorem 5. Off the phase transition curve,

$$\lim_{n \rightarrow \infty} \frac{\partial^2}{\partial \beta_2^2} \psi_n(\beta_1, \beta_2) = \frac{\partial^2}{\partial \beta_2^2} \lim_{n \rightarrow \infty} \psi_n(\beta_1, \beta_2) = \frac{p^2(x^*)^{2p-2}}{|\ell''(x^*)|}.$$

On the transition curve except at the critical point,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \frac{\partial^2}{\partial \beta_2^2} \psi_n(\beta_1, \beta_2) = \frac{((x_1^*)^p - (x_2^*)^p)^2 \sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}{\left(\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} + \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|} \right)^2}.$$

At the critical point,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \frac{\partial^2}{\partial \beta_2^2} \psi_n(\beta_1, \beta_2) = \frac{2\sqrt{6}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{(p-1)^{2p-1}}{p^{2p-\frac{3}{2}}}.$$

Theorem 6. Off the phase transition curve,

$$\lim_{n \rightarrow \infty} \frac{\partial^2}{\partial \beta_1 \partial \beta_2} \psi_n(\beta_1, \beta_2) = \frac{\partial^2}{\partial \beta_1 \partial \beta_2} \lim_{n \rightarrow \infty} \psi_n(\beta_1, \beta_2) = \frac{p(x^*)^{p-1}}{|\ell''(x^*)|}.$$

On the transition curve except at the critical point,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\partial^2}{\partial \beta_1 \partial \beta_2} \psi_n(\beta_1, \beta_2) \\ &= \frac{((x_1^*)^p - (x_2^*)^p)(x_1^* - x_2^*) \sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}{\left(\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} + \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|} \right)^2}. \end{aligned}$$

At the critical point,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} \frac{\partial^2}{\partial \beta_1 \partial \beta_2} \psi_n(\beta_1, \beta_2) = \frac{2\sqrt{6}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{(p-1)^p}{p^{p+\frac{1}{2}}}.$$

The next theorem gives the limiting edge densities.

Theorem 7. *Off the phase transition curve and at the critical point,*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n(X_{12} = 1) = 1 - \lim_{n \rightarrow \infty} \mathbb{P}_n(X_{12} = 0) = x^*. \quad (3.1)$$

On the phase transition curve except at the critical point,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n(X_{12} = 1) = 1 - \lim_{n \rightarrow \infty} \mathbb{P}_n(X_{12} = 0) = \alpha x_1^* + (1 - \alpha)x_2^*, \quad (3.2)$$

where

$$\alpha := \frac{\sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}{\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} + \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}. \quad (3.3)$$

In [18] it is proved that, off the phase transition curve and at the critical point, for large n a typical graph behaves like the Erdős-Rényi graph $G(n, p^*)$, where $p^* = x^*$ is the unique global maximizer of ℓ . (See Theorem 3.4 of [18] for a more precise statement.) It is also shown that, on the phase transition curve except at the critical point, for large n a typical graph behaves like $G(n, p^*)$, where p^* is a mixture of the two global maximizers $x_1^* < x_2^*$ of ℓ . However, p^* is not determined explicitly. In our model, since we consider only directed graphs, we do not obtain an Erdős-Rényi graph in the $n \rightarrow \infty$ limit. Nevertheless, Theorem 7 is a qualitatively similar result about limiting edge probabilities, with an explicit formula for the mixture of edge probabilities, $p^* = \alpha x_1^* + (1 - \alpha)x_2^*$, along the phase transition curve.

4. KEY ESTIMATES

First we have the following formula for the normalization $Z_n(\beta_1, \beta_2)$:

Proposition 8. *Let W be a binomial random variable with parameters n and $\frac{1}{2}$:*

$$\mathbb{P}(W = i) = 2^{-n} \binom{n}{i}.$$

Then

$$Z_n(\beta_1, \beta_2) = 2^{n^2} \left(\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \right)^n.$$

Next we approximate the expectation in Proposition 8 in terms of an integral:

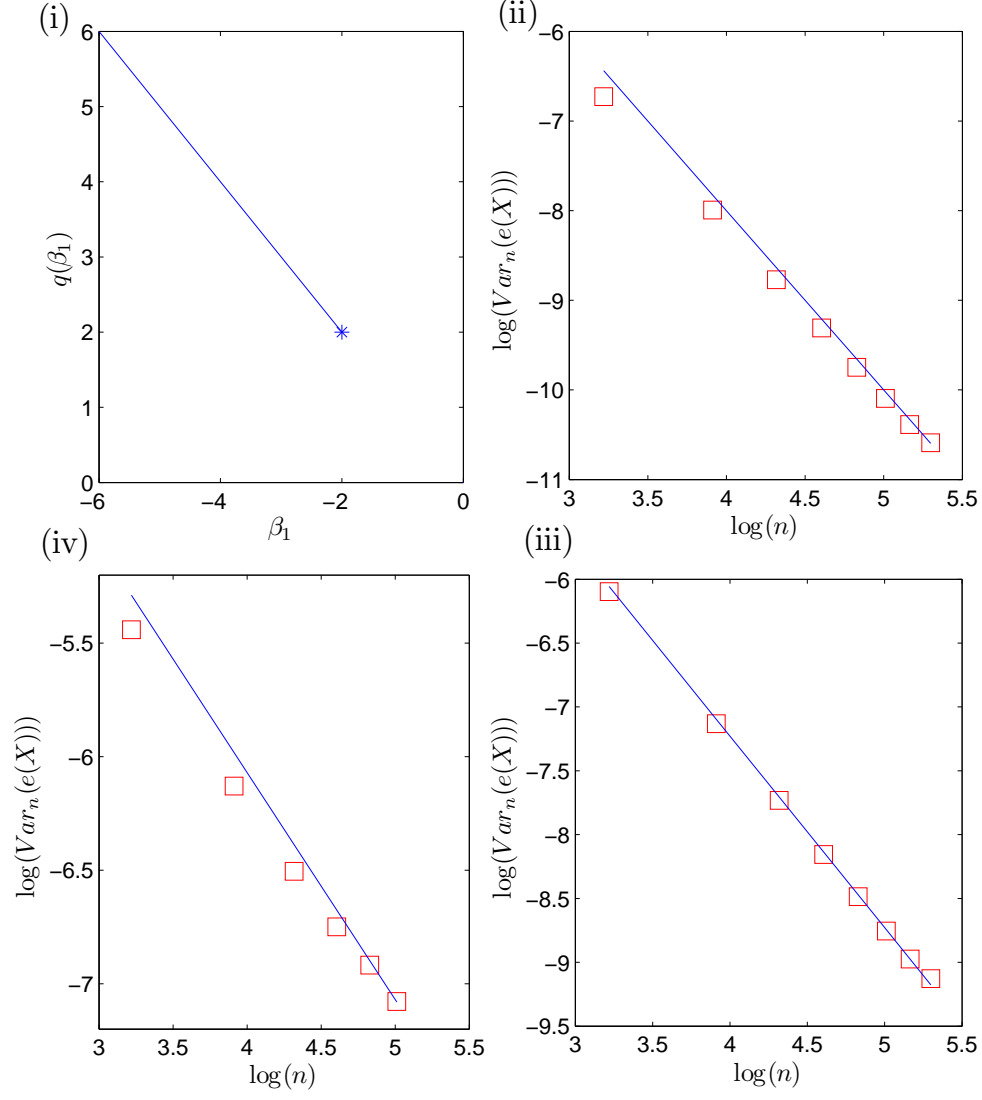


FIGURE 2. (i): The graph of the phase transition curve $\beta_2 = q(\beta_1)$ when $p = 2$, with the critical point labeled by *. (ii)-(iv): Scaling of the variance of $e(X)$ (ii) off the phase transition curve, (iii) at the critical point, and (iv) on the phase transition curve away from the critical point. For (ii)-(iv) we use $p = 2$ and (β_1, β_2) values of $(-3/2, 3/2)$, $(-2, 2)$ and $(-5/2, 5/2)$, respectively. The straight lines are obtained from the scaling in Theorem 4, and the squares are obtained by Monte Carlo simulation.

Proposition 9. *Let W be a binomial random variable with parameters n and $\frac{1}{2}$. Then for any $r < 1$,*

$$\begin{aligned} & \mathbb{E} \left[W^k \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \\ &= \begin{cases} (1 + O(n^{1/2-r})) n^k 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{x^{2k}}{x(1-x)}} e^{n\ell(x)} dx, & (\beta_1, \beta_2) \neq (\beta_1^c, \beta_2^c) \\ (1 + O(n^{1/4-r})) n^k 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{x^{2k}}{x(1-x)}} e^{n\ell(x)} dx, & (\beta_1, \beta_2) = (\beta_1^c, \beta_2^c) \end{cases} \end{aligned}$$

Lastly we give a technical lemma for computing the integral in Proposition 9:

Proposition 10. *Let f be an analytic function in $(0, 1)$ with Taylor expansion at $c \in (0, 1)$ given by*

$$f(x) = d_0(c) + d_1(c)(x - c) + d_2(c)(x - c)^2 + \dots, \quad d_j(c) := \frac{f^{(j)}(c)}{j!}.$$

For $c \in (0, 1)$, define

$$\begin{aligned} b_k(c) &= \frac{\ell^k(c)}{k!}, \\ \alpha_k(c) &= \frac{1}{2} \Gamma\left(\frac{k}{2}\right) |b_2(c)|^{-k/2}, \\ \gamma_k(c) &= \frac{1}{4} \Gamma\left(\frac{k}{4}\right) |b_4(c)|^{-k/4}. \end{aligned} \tag{4.1}$$

Assume that $f(x)e^{n\ell(x)} \in L^1[0, 1]$ for each n . Then as $n \rightarrow \infty$, we have the following.

(i) Off the phase transition curve,

$$\int_0^1 f(x) e^{n\ell(x)} dx = e^{n\ell(c)} \left[n^{-1/2} d_0 \alpha_1 + n^{-3/2} \Lambda + O(n^{-5/2}) \right]$$

where

$$\Lambda := d_2 \alpha_3 + d_1 b_3 \alpha_5 + d_0 b_4 \alpha_5 + \frac{1}{2} d_0 b_3^2 \alpha_7$$

with $c = x^*$ the unique maximizer of ℓ , and $d_j = d_j(c)$, $b_j = b_j(c)$, $\alpha_j = \alpha_j(c)$.

(ii) On the phase transition curve except at the critical point,

$$\int_0^1 f(x) e^{n\ell(x)} dx = e^{n\ell(c)} \left[n^{-1/2} (d_0(c_1) \alpha_1(c_1) + d_0(c_2) \alpha_1(c_2)) + O(n^{-3/2}) \right]$$

where c_1 and c_2 are the maximizers of ℓ .

(iii) At the critical point,

$$\int_0^1 f(x) e^{n\ell(x)} dx = e^{n\ell(c)} \left[n^{-1/4} d_0 \gamma_1 + n^{-3/4} \Theta + O(n^{-5/4}) \right]$$

where

$$\Theta := d_2 \gamma_3 + d_1 b_5 \gamma_7 + d_0 b_6 \gamma_7 + \frac{1}{2} d_0 b_5^2 \gamma_{11}$$

with $c = x^*$ the unique maximizer of ℓ , and $d_j = d_j(c)$, $b_j = b_j(c)$, $\gamma_j = \gamma_j(c)$.

Note that this strategy allows for a relatively precise computation of $Z_n(\beta_1, \beta_2)$. Unfortunately, arbitrary precision cannot be achieved, due to the error inherent in the sum to integral approximation of Proposition 9.

5. PROOFS

Before turning to the proofs of the theorems in Section 3, we will prove the estimates in Section 4. The following result will be needed in almost all of our proofs.

Proposition 11. *Off the phase transition curve,*

$$\ell'(x^*) = 0, \quad \ell''(x^*) < 0.$$

On the phase transition curve except at the critical point,

$$\ell'(x_1^*) = \ell'(x_2^*) = 0, \quad \ell''(x_1^*) < 0, \quad \ell''(x_2^*) < 0.$$

At the critical point,

$$\ell'(x^*) = \ell''(x^*) = \ell'''(x^*) = 0, \quad \ell^{(4)}(x^*) = \frac{-p^5}{(p-1)^2} < 0.$$

Proof. It is straightforward to compute that

$$\begin{aligned} \ell'(x) &= \beta_1 + p\beta_2 x^{p-1} - \log\left(\frac{x}{1-x}\right), \\ \ell''(x) &= p(p-1)\beta_2 x^{p-2} - \frac{1}{x} - \frac{1}{1-x}, \\ \ell'''(x) &= p(p-1)(p-2)\beta_2 x^{p-3} + \frac{1}{x^2} - \frac{1}{(1-x)^2}, \\ \ell^{(4)}(x) &= p(p-1)(p-2)(p-3)\beta_2 x^{p-4} - \frac{2}{x^3} - \frac{2}{(1-x)^3}. \end{aligned}$$

Since $\lim_{x \rightarrow 0^+} \ell'(x) = +\infty$ and $\lim_{x \rightarrow 1^-} \ell'(x) = -\infty$, the maximum is achieved at a local maximum, we have $\ell'(x^*) = 0$.

Let us first show that $\ell''(x^*) < 0$ off the critical point (where x^* denotes either x_1^* or x_2^* if we are on the phase transition curve). Following the proof of Proposition 3.2 in Radin and Yin [18], we first analyze the properties of $\ell''(x)$. We can re-write $\ell''(x)$ as

$$\ell''(x) = x^{p-2}p(p-1) \left[\beta_2 - \frac{1}{p(p-1)x^{p-1}(1-x)} \right]. \quad (5.1)$$

Consider the function

$$m(x) := \frac{1}{p(p-1)x^{p-1}(1-x)}. \quad (5.2)$$

It is easy to observe that $m(x) \geq \frac{p-1}{(p-1)^p}$ and the equality holds if and only if $x = \frac{p-1}{p}$.

(i) If $\beta_2 < \frac{p-1}{(p-1)^p}$, $\ell''(x) < 0$ on $[0, 1]$ and in particular $\ell''(x^*) < 0$.

(ii) If $\beta_2 > \frac{p-1}{(p-1)^p}$, there exist $0 < x_1 < \frac{p-1}{p} < x_2 < 1$ so that $\ell''(x) < 0$ on $0 < x < x_1$, $\ell''(x) > 0$ on $x_1 < x < x_2$ and $\ell''(x) < 0$ on $x_2 < x < 1$. Moreover $\ell''(x_1) = \ell''(x_2) = 0$. If $\ell'(x_1) \geq 0$, $\ell(x)$ has a unique local and hence global maximizer $x^* > x_2$; if $\ell'(x_2) \leq 0$, $\ell(x)$ has a unique local and hence global maximizer $x^* < x_1$. Finally, if $\ell'(x_1) < 0 < \ell'(x_2)$, then $\ell(x)$ has two local maximizers x_1^* and

x_2^* so that $x_1^* < x_1 < \frac{p-1}{p} < x_2 < x_2^*$. Since ℓ'' vanishes only at x_1 and x_2 , we have proved that $\ell''(x^*) < 0$.

(iii) If $\beta_2 = \frac{p^{p-1}}{(p-1)^p}$, $\ell''(x) \leq 0$ on $[0, 1]$ and $\ell''(x) = 0$ if and only if $x = \frac{p-1}{p}$ by the properties of $m(x)$. Therefore, $\ell''(x^*) = 0$ if and only if $x^* = \frac{p-1}{p}$. Since $\ell'(x^*) = 0$, $x^* = \frac{p-1}{p}$ if and only if

$$\beta_1 = -p \frac{p^{p-1}}{(p-1)^p} \left(\frac{p-1}{p} \right)^{p-1} + \log \left(\frac{\frac{p-1}{p}}{1 - \frac{p-1}{p}} \right) = \beta_1^c, \quad (5.3)$$

Hence $\ell''(x^*) < 0$ off the critical point and $\ell''(x^*) = 0$ at the critical point.

Furthermore, at the critical point $(\beta_1, \beta_2) = (\beta_1^c, \beta_2^c)$, we can compute that

$$\ell'''(u^*) = p(p-1)(p-2) \frac{p^{p-1}}{(p-1)^p} \frac{(p-1)^{p-3}}{p^{p-3}} + \frac{p^2}{(p-1)^2} - p^2 = 0.$$

Moreover,

$$\begin{aligned} \ell^{(4)}(u^*) &= p(p-1)(p-2)(p-3) \frac{p^{p-1}}{(p-1)^p} \frac{(p-1)^{p-4}}{p^{p-4}} - \frac{2p^3}{(p-1)^3} - 2p^3 \\ &= \frac{-p^5}{(p-1)^2} < 0. \end{aligned}$$

□

The next three proofs are for the results in Section 4.

Proof of Proposition 8. Let $Y = (Y_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ matrix of i.i.d. Bernoulli random variables:

$$\mathbb{P}(Y_{ij} = 0) = \frac{1}{2} = \mathbb{P}(Y_{ij} = 1).$$

For $i = 1, \dots, n$ define

$$W_i = \sum_{j=1}^n Y_{ij}.$$

Then

$$\begin{aligned} Z_n(\beta_1, \beta_2) &= 2^{n^2} \mathbb{E} [\exp (n^2(\beta_1 e(Y) + \beta_2 s(Y)))] \\ &= 2^{n^2} \mathbb{E} \left[\exp \left(\sum_{i=1}^n \beta_1 W_i + \frac{\beta_2}{n^{p-1}} W_i^p \right) \right] \\ &= 2^{n^2} \mathbb{E} \left[\prod_{i=1}^n \exp \left(\beta_1 W_i + \frac{\beta_2}{n^{p-1}} W_i^p \right) \right] \\ &= 2^{n^2} \prod_{i=1}^n \mathbb{E} \left[\exp \left(\beta_1 W_i + \frac{\beta_2}{n^{p-1}} W_i^p \right) \right] \\ &= 2^{n^2} \left(\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \right)^n. \end{aligned}$$

□

Proof of Proposition 9. We will prove only the case $k = 0$, as the other cases are easy extensions. Observe that

$$\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] = 2^{-n} \sum_{i=1}^n \binom{n}{i} \exp \left(\beta_1 i + \frac{\beta_2}{n^{p-1}} i^p \right).$$

Using the fact that for all $n \geq 1$,

$$n \log n - n + \frac{1}{2} \log n \leq \log n! \leq n \log n - n + \frac{1}{2} \log n + 1,$$

we obtain

$$\binom{n}{i} \leq \exp \left(n \left[-\frac{i}{n} \log \frac{i}{n} - \left(1 - \frac{i}{n} \right) \log \left(1 - \frac{i}{n} \right) + \frac{1}{2n} \log \frac{n}{i(n-i)} + \frac{1}{n} \right] \right). \quad (5.4)$$

Define

$$A_n = \{i \in \{1, \dots, n\} : i/n \in (\varepsilon, 1 - \varepsilon)\}$$

where $\varepsilon > 0$ will be specified momentarily. From (5.4), for any $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} \max_{i \in \{1, \dots, n\} \setminus A_n} \binom{n}{i} \exp \left(\beta_1 i + \frac{\beta_2}{n^{p-1}} i^p \right) &\leq e \left(1 - \frac{1}{n} \right)^{-\frac{1}{2n}} \sup_{x \in [0, 1] \setminus (\varepsilon, 1 - \varepsilon)} e^{n\ell(x)} \\ &\leq 3 \sup_{x \in [0, 1] \setminus (\varepsilon, 1 - \varepsilon)} e^{n\ell(x)}. \end{aligned}$$

Since $\ell'(x) \rightarrow \infty$ as $x \rightarrow 0$ and $\ell'(x) \rightarrow -\infty$ as $x \rightarrow 1$, we may choose $\varepsilon > 0$ such that for some $\delta > 0$,

$$\sup_{x \in [0, 1] \setminus (\varepsilon, 1 - \varepsilon)} \ell(x) < \ell(x^*) - \delta.$$

Thus,

$$\sum_{i \in \{1, \dots, n\} \setminus A_n} \binom{n}{i} \exp \left(\beta_1 i + \frac{\beta_2}{n^{p-1}} i^p \right) = O \left(e^{n(\ell(x^*) - \delta)} \right).$$

For $i \in A_n$, Stirling's formula allows us to write

$$\begin{aligned} \binom{n}{i} &= (1 + O(n^{-1})) \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{i(n-i)}} \\ &\quad \times \exp \left(n \left[-\frac{i}{n} \log \frac{i}{n} - \left(1 - \frac{i}{n} \right) \log \left(1 - \frac{i}{n} \right) \right] \right). \end{aligned}$$

The last two displays yield

$$\begin{aligned} &\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \\ &= 2^{-n} \left(\sum_{i=1}^n \binom{n}{i} \exp \left(\beta_1 i + \frac{\beta_2}{n^{p-1}} i^p \right) \right) \\ &= 2^{-n} \left(O \left(e^{n(\ell(x^*) - \delta)} \right) + \sum_{i \in A_n} \binom{n}{i} \exp \left(\beta_1 i + \frac{\beta_2}{n^{p-1}} i^p \right) \right) \\ &= 2^{-n} \left(O \left(e^{n(\ell(x^*) - \delta)} \right) + (1 + O(n^{-1})) \frac{1}{\sqrt{2\pi n}} \sum_{i \in A_n} \sqrt{\frac{1}{(i/n)(1-i/n)}} e^{n\ell(i/n)} \right). \end{aligned} \quad (5.5)$$

We will approximate the sum in (5.5) by an integral. Consider first the case off the transition curve. Thus, there is a unique maximizer x^* of ℓ , and $\ell'(x^*) = 0$, $\ell''(x^*) < 0$. Let $q \in (1/3, 1/2)$ and define

$$B_n = \{i \in \{1, \dots, n\} : i/n \in (x^* - n^{-q}, x^* + n^{-q})\}.$$

For any $j \in A_n$, note that

$$\begin{aligned} & \left| \frac{1}{n} \sqrt{\frac{1}{(j/n)(1-j/n)}} e^{n\ell(j/n)} - \int_{j/n}^{j/n+1/n} \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right| \\ & \leq \frac{1}{n} \max_{x,y \in [j/n, j/n+1/n]} \left| \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} - \sqrt{\frac{1}{y(1-y)}} e^{n\ell(y)} \right| \\ & \leq \frac{1}{n} \max_{x,y \in [j/n, j/n+1/n]} \sqrt{\frac{1}{x(1-x)}} \max_{x,y \in [j/n, j/n+1/n]} |e^{n\ell(x)} - e^{n\ell(y)}| \quad (5.6) \\ & \quad + \frac{1}{n} e^{n\ell(x^*)} \max_{x,y \in [j/n, j/n+1/n]} \left| \sqrt{\frac{1}{x(1-x)}} - \sqrt{\frac{1}{y(1-y)}} \right| \\ & = O(n^{-1}) \max_{x,y \in [j/n, j/n+1/n]} |e^{n\ell(x)} - e^{n\ell(y)}| + O(n^{-2}) e^{n\ell(x^*)}. \end{aligned}$$

Fix $j \in A_n$ and let $x, y \in [j/n, j/n+1/n]$. Note that for all x ,

$$|e^x - 1| \leq e^{|x|} - 1.$$

We use this, the fact that $\ell''(x^*) < 0$, and the mean value theorem to write

$$\begin{aligned} |e^{n\ell(x)} - e^{n\ell(y)}| &= e^{n\ell(x^*)} e^{n(\ell(y)-\ell(x^*))} |e^{n(\ell(x)-\ell(y))} - 1| \\ &= e^{n\ell(x^*)} \exp\left(n \frac{\ell''(x^*)}{2} (y-x^*)^2 + n \frac{\ell'''(\xi)}{6} (y-x^*)^3\right) \\ & \quad \times \left| \exp\left(n \ell'(y)(x-y) + \frac{n \ell''(\nu)}{2} (x-y)^2\right) - 1 \right| \\ &= e^{n\ell(x^*)} \exp\left(n \frac{\ell''(x^*)}{2} (y-x^*)^2 + n \frac{\ell'''(\xi)}{6} (y-x^*)^3\right) \\ & \quad \times \left| \exp\left(n \ell''(\zeta)(y-x^*)(x-y) + \frac{n \ell''(\nu)}{2} (x-y)^2\right) - 1 \right| \\ &\leq e^{n\ell(x^*)} \exp\left(n \frac{-|\ell''(x^*)|}{2} (y-x^*)^2 + n \frac{\ell'''(\xi)}{6} (y-x^*)^3\right) \\ & \quad \times \left(\exp\left(n |\ell''(\zeta)| |y-x^*| |x-y| + \frac{n |\ell''(\nu)|}{2} (x-y)^2\right) - 1 \right) \quad (5.7) \end{aligned}$$

where ξ and ζ are between y and x^* , and ν is between y and x . Observe that

$$\begin{aligned} & \exp\left(n \frac{-|\ell''(x^*)|}{2} (y-x^*)^2 + n \frac{\ell'''(\xi)}{6} (y-x^*)^3\right) \\ &= \begin{cases} O\left(\exp\left(-\frac{|\ell''(x^*)|}{2} n^{1-2q}\right)\right) (1 + O(n)), & j \notin B_n \\ 1 + O(n^{1-3q}), & j \in B_n \end{cases} \end{aligned}$$

and that

$$\begin{aligned} & \exp \left(n|\ell''(\zeta)||y - x^*||x - y| + \frac{n|\ell''(\nu)|}{2}(x - y)^2 \right) - 1 \\ &= \begin{cases} O(1), & j \notin B_n \\ O(n^{-q}), & j \in B_n \end{cases} \end{aligned}$$

Let $t = 1 - 2q > 0$ and $\omega \in (0, |\ell''(x^*)|/2)$. The last three displays show that

$$\max_{x, y \in [j/n, j/n+1/n]} \left| e^{n\ell(x)} - e^{n\ell(y)} \right| = \begin{cases} e^{n\ell(x^*)} O(\exp(-\omega n^t)), & j \notin B_n \\ e^{n\ell(x^*)} O(n^{-q}), & j \in B_n \end{cases},$$

and so from (5.6),

$$\begin{aligned} & \left| \frac{1}{n} \sqrt{\frac{1}{(j/n)(1-j/n)}} e^{n\ell(j/n)} - \int_{j/n}^{j/n+1/n} \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right| \\ &= \begin{cases} e^{n\ell(x^*)} O(\exp(-\omega n^t)), & j \notin B_n \\ e^{n\ell(x^*)} O(n^{-1-q}), & j \in B_n \end{cases}. \end{aligned} \quad (5.8)$$

Observe that

$$|B_n| = O(n^{1-q}), \quad |A_n \setminus B_n| = O(1). \quad (5.9)$$

Now from (5.8), for any $r < 1$,

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i \in A_n} \sqrt{\frac{1}{(i/n)(1-i/n)}} e^{n\ell(i/n)} - \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right| \\ & \leq e^{n\ell(x^*)} (|B_n| O(n^{-1-q}) + |A_n \setminus B_n| O(\exp(-\omega n^t))) \\ & \quad + \int_{\varepsilon+1/n}^{1-\varepsilon-1/n} \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \\ & \leq e^{n\ell(x^*)} (O(n^{-2q}) + O(\exp(-\omega n^t))) + O(e^{n(\ell(x^*)-\delta)}) \\ & \leq e^{n\ell(x^*)} O(n^{-r}). \end{aligned} \quad (5.10)$$

Now by (5.10) and Proposition 10,

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i \in A_n} \sqrt{\frac{1}{(i/n)(1-i/n)}} e^{n\ell(i/n)} - \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right| \\ & \times \left(\int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right)^{-1} = O(n^{1/2-r}). \end{aligned}$$

Thus,

$$\frac{1}{n} \sum_{i \in A_n} \sqrt{\frac{1}{(i/n)(1-i/n)}} e^{n\ell(i/n)} = \left(1 + O(n^{1/2-r}) \right) \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx.$$

Now from (5.5) we conclude

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \\ &= \left(1 + O \left(n^{1/2-r} \right) \right) 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx. \end{aligned}$$

Next, consider (β_1, β_2) on the transition curve away from the critical point. By Theorem 2, there are two maximizers of ℓ , say x_1^* and x_2^* . Defining

$$B_n = \{i \in \{1, \dots, n\} : i/n \in (x_1^* - n^{-q}, x_1^* + n^{-q}) \cup (x_2^* - n^{-q}, x_2^* + n^{-q})\},$$

it is not hard to see that the arguments above can be repeated to obtain the same result.

Finally, consider the case at the critical point. Here, equation (5.6) still holds, but (5.7) needs to be modified, as follows. By Proposition 11, we have $\ell'(x^*) = \ell''(x^*) = \ell'''(x^*) = 0$ and $\ell^{(4)}(x^*) < 0$, so by the mean value theorem we have

$$\begin{aligned} & \left| e^{n\ell(x)} - e^{n\ell(y)} \right| \\ &= e^{n\ell(x^*)} e^{n(\ell(y) - \ell(x^*))} \left| e^{n(\ell(x) - \ell(y))} - 1 \right| \\ &= e^{n\ell(x^*)} \exp \left(n \frac{\ell^{(4)}(x^*)}{4!} (y - x^*)^4 + n \frac{\ell^{(5)}(\xi)}{5!} (y - x^*)^5 \right) \\ & \quad \times \left| \exp \left(n\ell'(y)(x - y) + \frac{n\ell''(\nu)}{2} (x - y)^2 \right) - 1 \right| \\ &= e^{n\ell(x^*)} \exp \left(n \frac{\ell^{(4)}(x^*)}{4!} (y - x^*)^4 + n \frac{\ell^{(5)}(\xi)}{5!} (y - x^*)^5 \right) \\ & \quad \times \left| \exp \left(n\ell^{(4)}(\zeta)(v - x^*)(u - x^*)(y - x^*)(x - y) + \frac{n\ell''(\nu)}{2} (x - y)^2 \right) - 1 \right| \\ &\leq e^{n\ell(x^*)} \exp \left(n \frac{-|\ell^{(4)}(x^*)|}{4!} (y - x^*)^4 + n \frac{\ell^{(5)}(\xi)}{5!} (y - x^*)^5 \right) \\ & \quad \times \left(\exp \left(n|\ell^{(4)}(\zeta)||v - x^*||u - x^*||y - x^*||x - y| + \frac{n|\ell''(\nu)|}{2} (x - y)^2 \right) - 1 \right) \end{aligned} \tag{5.11}$$

where u, v, ξ and ζ are between y and x^* , and ν is between y and x . Let $q \in (1/5, 1/4)$ and note that

$$\begin{aligned} & \exp \left(n \frac{-|\ell^{(4)}(x^*)|}{4!} (y - x^*)^4 + n \frac{\ell^{(5)}(\xi)}{5!} (y - x^*)^5 \right) \\ &= \begin{cases} O \left(\exp \left(\frac{-|\ell^{(4)}(x^*)|}{4!} n^{1-4q} \right) \right) (1 + O(n)), & j \notin B_n \\ 1 + O(n^{1-5q}), & j \in B_n \end{cases} \end{aligned}$$

and that

$$\begin{aligned} & \exp \left(n|\ell^{(4)}(\zeta)||v - x^*||u - x^*||y - x^*||x - y| + \frac{n|\ell''(\nu)|}{2} (x - y)^2 \right) - 1 \\ &= \begin{cases} O(1), & j \notin B_n \\ O(n^{-3q}), & j \in B_n. \end{cases} \end{aligned}$$

Let $\omega \in (0, |\ell^{(4)}(x^*)|/4!)$ and $t = 1 - 4q > 0$. The last three displays show that

$$\max_{x,y \in [j/n, j/n+1/n]} |e^{n\ell(x)} - e^{n\ell(y)}| = \begin{cases} e^{n\ell(x^*)} O(\exp(-\omega n^t)), & j \notin B_n \\ e^{n\ell(x^*)} O(n^{-3q}), & j \in B_n \end{cases}.$$

So from (5.6),

$$\begin{aligned} & \left| \frac{1}{n} \sqrt{\frac{1}{(j/n)(1-j/n)}} e^{n\ell(j/n)} - \int_{j/n}^{j/n+1/n} \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right| \\ &= \begin{cases} e^{n\ell(x^*)} O(\exp(-\omega n^t)), & j \notin B_n \\ e^{n\ell(x^*)} O(n^{-1-3q}), & j \in B_n \end{cases}. \end{aligned} \quad (5.12)$$

Using (5.12) and (5.9), for any $r < 1$,

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i \in A_n} \sqrt{\frac{1}{(i/n)(1-i/n)}} e^{n\ell(i/n)} - \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right| \\ & \leq e^{n\ell(x^*)} (|B_n| O(n^{-1-3q}) + |A_n \setminus B_n| O(\exp(-\omega n^t))) \\ & \quad + \int_{\varepsilon+1/n}^{1-\varepsilon-1/n} \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \\ & \leq e^{n\ell(x^*)} (O(n^{-4q}) + O(\exp(-\omega n^t))) + O(e^{n(\ell(x^*)-\delta)}) \\ & \leq e^{n\ell(x^*)} O(n^{-r}). \end{aligned} \quad (5.13)$$

Now by (5.13) and Proposition 10,

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i \in A_n} \sqrt{\frac{1}{(i/n)(1-i/n)}} e^{n\ell(i/n)} - \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right| \\ & \times \left(\int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \right)^{-1} = O(n^{1/4-r}). \end{aligned}$$

Thus,

$$\frac{1}{n} \sum_{i \in A_n} \sqrt{\frac{1}{(i/n)(1-i/n)}} e^{n\ell(i/n)} = \left(1 + O(n^{1/4-r})\right) \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx.$$

Now from (5.5) we conclude

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \\ &= \left(1 + O(n^{1/4-r})\right) 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx. \end{aligned}$$

□

Proof of Proposition 10. We will prove only (i) and (iii), as (ii) is standard. We first consider (i). Note that for $b > 0$ and $k \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} x^k e^{-bx^2} dx = \begin{cases} 0, & k \text{ odd} \\ \frac{1}{2} \Gamma\left(\frac{k+1}{2}\right) b^{-\frac{k+1}{2}}, & k \text{ even} \end{cases}$$

So for any $\delta > 0$,

$$\begin{aligned} \int_{-\delta}^{\delta} u^k e^{-nbu^2} du &= n^{-\frac{k+1}{2}} \int_{-\delta n^{1/2}}^{\delta n^{1/2}} x^k e^{-bx^2} dx \\ &= n^{-\frac{k+1}{2}} \left(O(e^{-bn}) + \int_{-\infty}^{\infty} x^k e^{-bx^2} dx \right) \\ &= \begin{cases} 0, & k \text{ odd} \\ \frac{1}{2} \Gamma\left(\frac{k+1}{2}\right) (nb)^{-\frac{k+1}{2}} + O(e^{-bn}), & k \text{ even} \end{cases} \end{aligned}$$

Now let $c = x^*$ and $u = x - c$, and pick $0 < \delta < \min\{c, 1 - c\}$. We use Taylor expansions of x^k and $\ell(x)$ at c and of e^x at zero, along with Proposition 11, to compute

$$\begin{aligned} &\int_{c-\delta}^{c+\delta} f(x) e^{n\ell(x)} dx \\ &= \int_{-\delta}^{\delta} [d_0 + d_1 u + \dots] e^{n(b_0 + b_1 u + b_2 u^2 + \dots)} du \\ &= e^{n\ell(c)} \int_{-\delta}^{\delta} [d_0 + d_1 u + \dots] e^{nb_2 u^2 + nb_3 u^3 + \dots} du \\ &= e^{n\ell(c)} \int_{-\delta}^{\delta} [d_0 + d_1 u + \dots] \left[1 + (nb_3 u^3 + \dots) + \frac{1}{2} (nb_3 u^3 + \dots)^2 + \dots \right] e^{nb_2 u^2} du \\ &= e^{n\ell(c)} \left[n^{-1/2} d_0 \alpha_1 + n^{-3/2} \Lambda + O(n^{-5/2}) \right] \end{aligned}$$

where the last step is obtained by collecting terms of the same order, and the interchange of sum and integral is justified by dominated convergence theorem. Since $x^* = c$ is the unique global maximizer of ℓ , we conclude that for some $\varepsilon > 0$,

$$\int_0^1 f(x) e^{n\ell(x)} dx = \int_{c-\delta}^{c+\delta} f(x) e^{n\ell(x)} dx + O\left(e^{n(\ell(c)-\varepsilon)}\right).$$

It follows that

$$\int_0^1 f(x) e^{n\ell(x)} dx = e^{n\ell(c)} \left[n^{-1/2} d_0 \alpha_1 + n^{-3/2} \Lambda + O(n^{-5/2}) \right].$$

Now we turn to (iii). Note that for $b > 0$ and $k \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} x^k e^{-bx^4} dx = \begin{cases} 0, & k \text{ odd} \\ \frac{1}{4} \Gamma\left(\frac{k+1}{4}\right) b^{-\frac{k+1}{4}}, & k \text{ even} \end{cases}.$$

So for any $\delta > 0$,

$$\begin{aligned} \int_{-\delta}^{\delta} u^k e^{-nbu^4} du &= n^{-\frac{k+1}{4}} \int_{-\delta n^{1/4}}^{\delta n^{1/4}} x^k e^{-bx^4} dx \\ &= n^{-\frac{k+1}{4}} \left(O(e^{-bn}) + \int_{-\infty}^{\infty} x^k e^{-bx^4} dx \right) \\ &= \begin{cases} 0, & k \text{ odd} \\ \frac{1}{4} \Gamma\left(\frac{k+1}{4}\right) (nb)^{-\frac{k+1}{4}} + O(e^{-bn}), & k \text{ even} \end{cases} \end{aligned}$$

As before we let $c = x^*$ and $u = x - c$, pick $0 < \delta = \min\{c, 1 - c\}$ and use Taylor expansions of x^k and $\ell(x)$ at c and e^x at zero, along with Proposition 11, to write

$$\begin{aligned}
& \int_{c-\delta}^{c+\delta} f(x) e^{n\ell(x)} dx \\
&= \int_{c-\delta}^{c+\delta} [d_0 + d_1 u + \dots] e^{n(b_0 + b_1 u + b_2 u^2 + \dots)} du \\
&= e^{n\ell(c)} \int_{c-\delta}^{c+\delta} [d_0 + d_1 u + \dots] e^{nb_4 u^4 + nb_5 u^5 + \dots} du \\
&= e^{n\ell(c)} \int_{c-\delta}^{c+\delta} [d_0 + d_1 u + \dots] \left[1 + (nb_5 u^5 + \dots) + \frac{1}{2}(nb_5 u^5 + \dots)^2 + \dots \right] e^{nb_4 u^4} du \\
&= e^{n\ell(c)} \left[n^{-1/4} d_0 \gamma_1 + n^{-3/4} \Theta + O(n^{-5/4}) \right],
\end{aligned}$$

where again the last step is obtained by collecting terms of the same order, and the interchange of sum and integral is justified by dominated convergence theorem. As before, since $x^* = c$ is the unique global maximizer of ℓ , we can conclude that

$$\int_0^1 f(x) e^{n\ell(x)} dx = e^{n\ell(c)} \left[n^{-1/4} d_0 \gamma_1 + n^{-3/4} \Theta + O(n^{-5/4}) \right].$$

□

The remainder of the proofs are for the results in Section 3.

Proof of Theorem 1. By Propositions 9 and 10, we have

$$\begin{aligned}
\psi_n(\beta_1, \beta_2) &= n^{-2} \log Z_n(\beta_1, \beta_2) \\
&= \log 2 + n^{-1} \log \mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \\
&= O(n^{-1} \log n) + \frac{1}{n} \log \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx \\
&= O(n^{-1} \log n) + \ell(x^*).
\end{aligned} \tag{5.14}$$

□

Proof of Theorem 3. (i) Along the phase transition curve, we have

$$\beta_1 + pq(\beta_1)(x_1^*)^{p-1} - \log \left(\frac{x_1^*}{1 - x_1^*} \right) = 0, \tag{5.15}$$

$$\beta_1 + pq(\beta_1)(x_2^*)^{p-1} - \log \left(\frac{x_2^*}{1 - x_2^*} \right) = 0, \tag{5.16}$$

$$\begin{aligned}
& \beta_1 x_1^* + q(\beta_1)(x_1^*)^p + x_1^* \log x_1^* + (1 - x_1^*) \log(1 - x_1^*) \\
&= \beta_1 x_2^* + q(\beta_1)(x_2^*)^p + x_2^* \log x_2^* + (1 - x_2^*) \log(1 - x_2^*).
\end{aligned} \tag{5.17}$$

Let $x_1^* < x_2^*$ be the two local maximizers of ℓ in the V-shaped region [18] that contains the phase transition curve except the critical point. By Proposition 11, $\ell''(x_1^*)$ and $\ell''(x_2^*)$ are nonzero away from the critical point. The implicit function theorem implies that then x_1^* and x_2^* are analytic functions of both β_1 and β_2 .

Differentiating (5.17) with respect to β_1 and using (5.15) and (5.16), we can show that

$$x_1^* + q'(\beta_1)(x_1^*)^p = x_2^* + q'(\beta_1)(x_2^*)^p,$$

which implies that $q'(\beta_1) = -\frac{x_1^* - x_2^*}{(x_1^*)^p - (x_2^*)^p}$. As $\beta_1 \rightarrow \beta_1^c$, $x_2^* - x_1^* \rightarrow 0$ and both x_2^* and x_1^* converge to the common maximizer $x_c^* = \frac{p-1}{p}$. Therefore,

$$\lim_{\beta_1 \rightarrow \beta_1^c} q'(\beta_1) = -\frac{1}{p(x_c^*)^{p-1}} = -\frac{p^{p-2}}{(p-1)^{p-1}}.$$

Since $x_1^* \rightarrow 0$ and $x_2^* \rightarrow 1$ as $\beta_1 \rightarrow -\infty$, we get $\lim_{\beta_1 \rightarrow -\infty} q'(\beta_1) = -1$.

(ii) Differentiating $q'(\beta_1)$ with respect to β_1 , we get

$$\begin{aligned} q''(\beta_1) = & -\frac{1}{((x_1^*)^p - (x_2^*)^p)^2} [(1-p)(x_1^*)^p + p(x_1^*)^{p-1}x_2^* - (x_2^*)^p] \frac{\partial x_1^*}{\partial \beta_1} \\ & - \frac{1}{((x_1^*)^p - (x_2^*)^p)^2} [(1-p)(x_2^*)^p + p(x_2^*)^{p-1}x_1^* - (x_1^*)^p] \frac{\partial x_2^*}{\partial \beta_1}. \end{aligned} \quad (5.18)$$

Differentiating (5.15) and (5.16) with respect to β_1 , we get

$$1 + pq'(\beta_1)(x_1^*)^{p-1} + \left[pq(\beta_1)(p-1)(x_1^*)^{p-2} - \frac{1}{x_1^*(1-x_1^*)} \right] \frac{\partial x_1^*}{\partial \beta_1} = 0, \quad (5.19)$$

$$1 + pq'(\beta_1)(x_2^*)^{p-1} + \left[pq(\beta_1)(p-1)(x_2^*)^{p-2} - \frac{1}{x_2^*(1-x_2^*)} \right] \frac{\partial x_2^*}{\partial \beta_1} = 0. \quad (5.20)$$

Notice that

$$1 + pq'(\beta_1)(x_1^*)^{p-1} = 1 - p \frac{x_1^* - x_2^*}{(x_1^*)^p - (x_2^*)^p} (x_1^*)^{p-1} > 0, \quad (5.21)$$

and

$$1 + pq'(\beta_1)(x_2^*)^{p-1} = 1 - p \frac{x_1^* - x_2^*}{(x_1^*)^p - (x_2^*)^p} (x_2^*)^{p-1} < 0, \quad (5.22)$$

Moreover, in Proposition 11, we showed that

$$\ell''(x_1^*) = pq(\beta_1)(p-1)(x_1^*)^{p-2} - \frac{1}{x_1^*(1-x_1^*)} < 0, \quad (5.23)$$

$$\ell''(x_2^*) = pq(\beta_1)(p-1)(x_2^*)^{p-2} - \frac{1}{x_2^*(1-x_2^*)} < 0. \quad (5.24)$$

Therefore, from (5.19), (5.20), (5.21), (5.22), (5.23) and (5.24), we conclude that $\frac{\partial x_1^*}{\partial \beta_1} > 0$ and $\frac{\partial x_2^*}{\partial \beta_1} < 0$. Finally, by noticing that in (5.18),

$$(1-p)(x_1^*)^p + p(x_1^*)^{p-1}x_2^* - (x_2^*)^p < 0,$$

$$(1-p)(x_2^*)^p + p(x_2^*)^{p-1}x_1^* - (x_1^*)^p > 0,$$

we conclude that $q''(\beta_1) > 0$. \square

In the proofs below, let $d_m^{(n)}$ be defined as in Proposition 10 for the function

$$f(x) = \frac{x^n}{\sqrt{x(1-x)}}.$$

Proof of Theorem 4. Off the phase transition curve, the result follows immediately from Theorem 1 and results in [18]. Thus, we prove only the last two displays in Theorem 4.

From the second line of (5.14), we have

$$\begin{aligned} \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) = n^{-1} & \left\{ \frac{\mathbb{E} \left[W^2 \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \right. \\ & \left. - \left(\frac{\mathbb{E} \left[W \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \right)^2 \right\}. \end{aligned} \quad (5.25)$$

We use Proposition 9 and Proposition 10 to estimate each of the terms in (5.25).

We first consider the case on the transition curve excluding the critical point. By Theorem 2, there are two global maximizers $x_1^* < x_2^*$ of ℓ . Let us write $\ell(x_1^*) = \ell(x_2^*) = \ell(x^*)$. By Proposition 10 and Proposition 9, for any $r < 1$, we have

$$\begin{aligned} & \mathbb{E} \left[W^k \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \\ &= \left[1 + O(n^{\frac{1}{2}-r}) \right] \frac{n^k 2^{-n} \sqrt{n}}{\sqrt{2\pi}} \int_0^1 \sqrt{\frac{x^{2k}}{x(1-x)}} e^{n\ell(x)} dx \\ &= \left[1 + O(n^{\frac{1}{2}-r}) \right] \frac{n^k 2^{-n} \sqrt{n}}{\sqrt{2\pi}} \frac{e^{n\ell(x^*)}}{\sqrt{n}} \left[\frac{\sqrt{\frac{(x_1^*)^{2k}}{x_1^*(1-x_1^*)}}}{\sqrt{2\pi\ell''(x_1^*)}} + \frac{\sqrt{\frac{(x_2^*)^{2k}}{x_2^*(1-x_2^*)}}}{\sqrt{2\pi\ell''(x_2^*)}} + O(n^{-1}) \right] \\ &= \frac{n^k 2^{-n} e^{n\ell(x^*)}}{2\pi} \left[\frac{(x_1^*)^k}{\sqrt{x_1^*(1-x_1^*)\ell''(x_1^*)}} + \frac{(x_2^*)^k}{\sqrt{x_2^*(1-x_2^*)\ell''(x_2^*)}} + O(n^{\frac{1}{2}-r}) \right]. \end{aligned} \quad (5.26)$$

Hence,

$$\begin{aligned} & \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) \\ &= n^{-1} n^2 \frac{\frac{(x_1^*)^2}{\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \frac{(x_2^*)^2}{\sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}}{\frac{1}{\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \frac{1}{\sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}} \\ & \quad - n^{-1} n^2 \frac{\left(\frac{x_1^*}{\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \frac{x_2^*}{\sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}} \right)^2}{\left(\frac{1}{\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \frac{1}{\sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}} \right)^2} + O(n^{\frac{3}{2}-r}) \\ &= n \frac{\frac{(x_1^* - x_2^*)^2}{\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}}{\left(\frac{1}{\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \frac{1}{\sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}} \right)^2} + O(n^{\frac{3}{2}-r}) \\ &= n \frac{(x_1^* - x_2^*)^2 \sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}{\left(\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} + \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|} \right)^2} + O(n^{\frac{3}{2}-r}). \end{aligned} \quad (5.27)$$

Next consider the case at the critical point. By Proposition 10 and Proposition 9, for any $r < 1$,

$$\begin{aligned} & \mathbb{E} \left[W^k \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \\ &= \left[1 + O(n^{\frac{1}{4}-r}) \right] \frac{n^k 2^{-n} \sqrt{n}}{\sqrt{2\pi}} \int_0^1 \sqrt{\frac{x^{2k}}{x(1-x)}} e^{n\ell(x)} dx \\ &= \frac{n^k 2^{-n} \sqrt{n}}{\sqrt{2\pi}} e^{n\ell(x^*)} \left[n^{-1/4} d_0^{(k)} \gamma_1 + n^{-3/4} \Theta^{(k)} + O(n^{-r}) \right], \end{aligned} \quad (5.28)$$

where

$$\Theta^{(k)} := d_2^{(k)} \gamma_3 + d_1^{(k)} b_5 \gamma_7 + d_0^{(k)} b_6 \gamma_7 + \frac{1}{2} d_0^{(k)} b_5^2 \gamma_{11}, \quad k = 0, 1, 2.$$

Then

$$\begin{aligned} d_0^{(0)} &= \frac{1}{\sqrt{x^*(1-x^*)}}, & d_0^{(1)} &= \frac{x^*}{\sqrt{x^*(1-x^*)}}, & d_0^{(2)} &= \frac{(x^*)^2}{\sqrt{x^*(1-x^*)}}, \\ d_1^{(0)} &= \frac{x^* - \frac{1}{2}}{(x^*(1-x^*))^{3/2}}, & d_1^{(1)} &= \frac{\frac{x^*}{2}}{(x^*(1-x^*))^{3/2}}, & d_1^{(2)} &= \frac{\frac{3}{2}(x^*)^2 - (x^*)^3}{(x^*(1-x^*))^{3/2}}, \\ d_2^{(0)} &= \frac{2(x^*)^2 - 2x^* + \frac{3}{4}}{2(x^*(1-x^*))^{5/2}}, & d_2^{(1)} &= \frac{(x^*)^2 - \frac{x^*}{4}}{2(x^*(1-x^*))^{5/2}}, & d_2^{(2)} &= \frac{\frac{3}{4}(x^*)^2}{2(x^*(1-x^*))^{5/2}}. \end{aligned}$$

It is easy to observe that $d_0^{(2)} d_0^{(0)} = (d_0^{(1)})^2$. By differentiating this identity, we get $d_1^{(2)} d_0^{(0)} + d_0^{(2)} d_1^{(0)} = 2d_1^{(1)} d_0^{(1)}$. Therefore, by (5.25) and (5.26),

$$\begin{aligned} & \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) \\ &= n^{-1} n^2 \frac{n^{-\frac{1}{4}} d_0^{(2)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(2)}}{n^{-\frac{1}{4}} d_0^{(0)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(0)}} \\ & \quad - n^{-1} n^2 \frac{(n^{-\frac{1}{4}} d_0^{(1)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(1)})^2}{(n^{-\frac{1}{4}} d_0^{(0)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(0)})^2} + O(n^{\frac{5}{4}-r}) \\ &= n \frac{n^{-1} \gamma_1 [d_0^{(2)} \Theta^{(0)} + d_0^{(0)} \Theta^{(2)} - 2d_0^{(1)} \Theta^{(1)}] + O(n^{-\frac{3}{2}})}{n^{-\frac{1}{2}} (d_0^{(0)})^2 \gamma_1^2} + O(n^{\frac{5}{4}-r}) \\ &= \frac{n^{\frac{1}{2}}}{(d_0^{(0)})^2 \gamma_1} \left[\gamma_3 \left(d_0^{(2)} d_2^{(0)} + d_0^{(0)} d_2^{(2)} - 2d_0^{(1)} d_2^{(1)} \right) \right] \\ & \quad + \frac{n^{\frac{1}{2}}}{(d_0^{(0)})^2 \gamma_1} \left[b_5 \gamma_7 \left(d_0^{(2)} d_1^{(0)} + d_0^{(0)} d_1^{(2)} - 2d_0^{(1)} d_1^{(1)} \right) \right] + O(n^{\frac{5}{4}-r}) \\ &= \frac{n^{\frac{1}{2}} \gamma_3}{(d_0^{(0)})^2 \gamma_1} \left(d_0^{(2)} d_2^{(0)} + d_0^{(0)} d_2^{(2)} - 2d_0^{(1)} d_2^{(1)} \right) + O(n^{\frac{5}{4}-r}) \\ &= n^{\frac{1}{2}} \frac{\gamma_3}{\gamma_1} + O(n^{\frac{5}{4}-r}) \\ &= n^{\frac{1}{2}} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{1}{\sqrt{\frac{\ell^{(4)}(x^*)}{4!}}} + O(n^{\frac{5}{4}-r}) = n^{\frac{1}{2}} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{2\sqrt{6}(p-1)}{p^{5/2}} + O(n^{\frac{5}{4}-r}), \end{aligned} \quad (5.29)$$

where we used Proposition 11 in the last line. \square

Proof of Theorem 5. We prove only the last two displays in Theorem 5, since the first display follows immediately from Theorem 1 and results in [18]. From the second line of (5.14), we have

$$\begin{aligned} \frac{\partial^2}{\partial \beta_2^2} \psi_n(\beta_1, \beta_2) = n^{-1} & \left\{ \frac{\mathbb{E} \left[\frac{W^{2p}}{n^{2(p-1)}} \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \right. \\ & \left. - \left(\frac{\mathbb{E} \left[\frac{W^p}{n^{p-1}} \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \right)^2 \right\}. \end{aligned} \quad (5.30)$$

Consider first the case on the phase transition curve excluding the critical point. Then, similar to the proof of Theorem 4, for any $r < 1$,

$$\begin{aligned} \frac{\partial^2}{\partial \beta_2^2} \psi_n(\beta_1, \beta_2) = n & \frac{((x_1^*)^p - (x_2^*)^p)^2 \sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}{\left(\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} + \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|} \right)^2} \\ & + O(n^{\frac{3}{2}-r}). \end{aligned}$$

Now consider the case at the critical point. We have

$$\begin{aligned} d_0^{(p)} &= \frac{(x^*)^p}{\sqrt{x^*(1-x^*)}}, \\ d_1^{(p)} &= \frac{(p - \frac{1}{2})(x^*)^p - (p-1)(x^*)^{p+1}}{(x^*(1-x^*))^{3/2}}, \\ d_2^{(p)} &= \frac{(p^2 - 2p + \frac{3}{4})(x^*)^p - (2p^2 - 5p + 2)(x^*)^{p+1} + (p^2 - 3p + 2)(x^*)^{p+2}}{2(x^*(1-x^*))^{5/2}}. \end{aligned}$$

It is easy to observe that $d_0^{(2p)} d_0^{(0)} = (d_0^{(p)})^2$. By differentiating this identity, we get $d_1^{(2p)} d_0^{(0)} + d_0^{(2p)} d_1^{(0)} = 2d_1^{(p)} d_0^{(p)}$. Similar to the proof of Theorem 4, for any $r < 1$,

$$\begin{aligned} & \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) \\ &= n \frac{(n^{-\frac{1}{4}} d_0^{(2p)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(2p)}) (n^{-\frac{1}{4}} d_0^{(0)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(0)}) - (n^{-\frac{1}{4}} d_0^{(p)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(p)})^2}{(n^{-\frac{1}{4}} d_0^{(0)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(0)})^2} \\ & \quad + O(n^{\frac{5}{4}-r}) \\ &= \frac{n^{\frac{1}{2}} \gamma_3}{(d_0^{(0)})^2 \gamma_1} \left(d_0^{(2p)} d_2^{(0)} + d_0^{(0)} d_2^{(2p)} - 2d_0^{(p)} d_2^{(p)} \right) + O(n^{\frac{5}{4}-r}) \\ &= p^2 (x^*)^{2p-2} \frac{\gamma_3}{\gamma_1} n^{1/2} + O(n^{\frac{5}{4}-r}) \\ &= n^{\frac{1}{2}} p^2 \left(\frac{p-1}{p} \right)^{2p-2} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{2\sqrt{6}(p-1)}{p^{5/2}} + O(n^{\frac{5}{4}-r}). \end{aligned}$$

\square

Proof of Theorem 6. Again we prove only the last two displays in the theorem. From the second line of (5.14), we have

$$\begin{aligned} & \frac{\partial^2}{\partial \beta_1 \partial \beta_2} \psi_n(\beta_1, \beta_2) \\ &= n^{-1} \frac{\mathbb{E} \left[W \frac{W^p}{n^{(p-1)}} \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \\ & \quad - \frac{\mathbb{E} \left[W \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \mathbb{E} \left[\frac{W^p}{n^{p-1}} \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\left(\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right] \right)^2}. \end{aligned} \quad (5.31)$$

Similar to the proof of Theorem 4, on the phase transition curve excluding the critical point, for any $r < 1$,

$$\begin{aligned} & \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) \\ &= n \frac{((x_1^*)^p - (x_2^*)^p)(x_1^* - x_2^*) \sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}{\left(\sqrt{x_1^*(1-x_1^*)|\ell''(x_1^*)|} + \sqrt{x_2^*(1-x_2^*)|\ell''(x_2^*)|} \right)^2} + O(n^{\frac{3}{2}-r}). \end{aligned}$$

Consider now the case at the critical point. It is easy to observe that $d_0^{(p+1)} d_0^{(0)} = (d_0^{(1)})(d_0^{(p)})$. By differentiating this identity, we get $d_1^{(p+1)} d_0^{(0)} + d_0^{(p+1)} d_1^{(0)} = d_1^{(1)} d_0^{(p)} + d_0^{(1)} d_1^{(p)}$. Therefore, similar to the proof of Theorem 4, we get for any $r < 1$,

$$\begin{aligned} & \frac{\partial^2}{\partial \beta_1^2} \psi_n(\beta_1, \beta_2) \\ &= n \frac{(n^{-\frac{1}{4}} d_0^{(p+1)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(p+1)})(n^{-\frac{1}{4}} d_0^{(0)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(0)})}{(n^{-\frac{1}{4}} d_0^{(0)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(0)})^2} \\ & \quad - n \frac{(n^{-\frac{1}{4}} d_0^{(1)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(1)})(n^{-\frac{1}{4}} d_0^{(p)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(p)})}{(n^{-\frac{1}{4}} d_0^{(0)} \gamma_1 + n^{-\frac{3}{4}} \Theta^{(0)})^2} + O(n^{\frac{5}{4}-r}) \\ &= n \frac{n^{-1} \gamma_1 [d_0^{(p+1)} \Theta^{(0)} + d_0^{(0)} \Theta^{(p+1)} - d_0^{(1)} \Theta^{(p)} - d_0^{(p)} \Theta^{(1)}] + O(n^{-\frac{3}{2}})}{n^{-\frac{1}{2}} (d_0^{(0)})^2 \gamma_1^2 + O(n^{-1})} + O(n^{\frac{5}{4}-r}) \\ &= \frac{n^{\frac{1}{2}} \gamma_3}{(d_0^{(0)})^2 \gamma_1} \left(d_0^{(p+1)} d_2^{(0)} + d_0^{(0)} d_2^{(p+1)} - d_0^{(1)} d_2^{(p)} - d_0^{(p)} d_2^{(1)} \right) + O(n^{\frac{5}{4}-r}) \\ &= p(x^*)^{p-1} \frac{\gamma_3}{\gamma_1} n^{1/2} + O(n^{\frac{5}{4}-r}) \\ &= p \left(\frac{p-1}{p} \right)^{p-1} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{2\sqrt{6}(p-1)}{p^{5/2}} n^{1/2} + O(n^{\frac{5}{4}-r}). \end{aligned} \quad (5.32)$$

□

Proof of Theorem 7. Observe first that $\mathbb{P}_n(X_{12} = 1) = \mathbb{E}_n[X_{12}] = \frac{1}{n} \mathbb{E}_n[\sum_{j=1}^n X_{1j}]$. Thus, off the transition curve we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{P}_n(X_{12} = 1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_n \left[\sum_{j=1}^n X_{1j} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\mathbb{E} \left[W \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \\
&= \lim_{n \rightarrow \infty} \frac{(1 + O(n^{1/2-4q})) 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{x^2}{x(1-x)}} e^{n\ell(x)} dx}{(1 + O(n^{1/2-4q})) 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx} \\
&= \lim_{n \rightarrow \infty} \frac{\sqrt{\frac{2\pi(x^*)^2}{x^*(1-x^*)|\ell''(x^*)|}} n^{-\frac{1}{2}} e^{n\ell(x^*)}}{\sqrt{\frac{2\pi}{x^*(1-x^*)|\ell''(x^*)|}} n^{-\frac{1}{2}} e^{n\ell(x^*)}} \\
&= x^*.
\end{aligned}$$

Similarly, at the critical point,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{P}_n(X_{12} = 1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\mathbb{E} \left[W \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \\
&= \lim_{n \rightarrow \infty} \frac{(1 + O(n^{1/4-4q})) 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{x^2}{x(1-x)}} e^{n\ell(x)} dx}{(1 + O(n^{1/4-4q})) 2^{-n} \sqrt{\frac{n}{2\pi}} \int_0^1 \sqrt{\frac{1}{x(1-x)}} e^{n\ell(x)} dx} \\
&= \lim_{n \rightarrow \infty} \frac{e^{n\ell(x^*)} n^{-\frac{1}{4}} d_0^{(1)} \gamma_1}{e^{n\ell(x^*)} n^{-\frac{1}{4}} d_0^{(0)} \gamma_1} \\
&= x^*.
\end{aligned}$$

Finally, on the phase transition curve except at the critical point,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{P}_n(X_{12} = 1) &= \lim_{n \rightarrow \infty} \frac{1}{n} \frac{\mathbb{E} \left[W \exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]}{\mathbb{E} \left[\exp \left(\beta_1 W + \frac{\beta_2}{n^{p-1}} W^p \right) \right]} \\
&= \lim_{n \rightarrow \infty} \frac{\left(\sqrt{\frac{2\pi(x_1^*)^2}{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \sqrt{\frac{2\pi(x_2^*)^2}{x_2^*(1-x_2^*)|\ell''(x_2^*)|}} \right) n^{-\frac{1}{2}} e^{n\ell(x^*)}}{\left(\sqrt{\frac{2\pi}{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \sqrt{\frac{2\pi}{x_2^*(1-x_2^*)|\ell''(x_2^*)|}} \right) n^{-\frac{1}{2}} e^{n\ell(x^*)}} \\
&= \frac{x_1^* \sqrt{\frac{1}{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + x_2^* \sqrt{\frac{1}{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}}{\sqrt{\frac{1}{x_1^*(1-x_1^*)|\ell''(x_1^*)|}} + \sqrt{\frac{1}{x_2^*(1-x_2^*)|\ell''(x_2^*)|}}}.
\end{aligned}$$

□

ACKNOWLEDGEMENTS

The authors are very grateful to Mei Yin for helpful discussions.

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